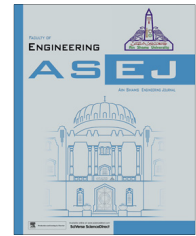




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The homotopy analysis method for q -difference equations

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Abstract The q -difference equations are kind of important problems in q -calculus and applied mathematics. In this paper, the homotopy analysis method is extended to find approximate solution for some of q -differential equations. The q -diffusion equation and some examples are analytically investigated. The series solutions obtained by the proposed method are checked by reducing the solutions of q -calculus problems to h -calculus approximate solutions when $q \rightarrow 1$.

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1. Introduction

At the last quarter of 20th century, q -calculus appears connections between physics and mathematics [1–9]. It has a lot of applications in different applied science and various mathematical areas, such as statistic physics fractal geometry, combinatorics, number theory, orthogonal polynomials, basic hypergeometric functions, quantum theory and theory of relativity. The q -differential equations are used to modelling the linear and nonlinear problems and played an important role in different fields of engineering and science. Recently, the semi-analytic techniques have been successfully employed to solve linear and nonlinear q -difference models, such as the differential transformation method [10–12], successive approxi-

ation method [13,14], variational iteration method [15,16] and homotopy perturbation method [17] and other methods. The homotopy analysis method (HAM) [18–27] is one of the semi-analytical techniques used most often for solving various differential equations in h -calculus; in this study, the homotopy analysis method is extended to solve q -differential equations. The solutions obtained by the proposed method are the semi-analytic solutions for the problems in h -calculus for the parameter $q \rightarrow 1$. We describe some definitions and properties of the q -calculus when $0 < q < 1$, let T_b be the timescales. $T_b = \{q^m : m \in \mathbb{Z}\} \cup \{0\}$. The q -number corresponding to the ordinary number k is defined as [28],

$$[k]_q = \frac{1 - q^n}{1 - q} = 1 + q + \dots + q^{n-1}, \quad (1)$$

so that k is the limit of $[k]_q$ as $q \rightarrow 1$ and the q -binomial coefficients are defined by:

$$\begin{bmatrix} k \\ i \end{bmatrix}_q = \frac{[k]_q!}{[i]_q! [k-i]_q!} \quad (2)$$

where $[k]_q! = [1]_q [2]_q \dots [k]_q$.

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Definition 1. The q -derivative of a real continuous function $f(x)$ is defined as follows [28]:

$$\frac{d_q}{d_q x} f(x) = \frac{f(qx) - f(x)}{(q-1)x}, x \neq 0, \quad q \in (0, 1). \quad (3)$$

When $q \rightarrow 1$ it reduces to the standard derivative $\frac{d}{dx} f(x) \rightarrow f'(x)$. Using the Definition 1 one can easily see that the effect of q -derivative on the power function is as follows:

$$\frac{d_q}{d_q x} (x-a)^k = [k]_q (x-a)^{k-1}, \quad (x, a \in \mathbb{R}, k \in \mathbb{N}).$$

Definition 2. The partial q -derivative of a function $f(x_1; x_2)$ to a variable x_1 is defined by [10]

$$\frac{\partial_q}{\partial_q x_1} f(x_1; x_2) = \frac{f(qx_1; x_2) - f(x_1; x_2)}{(q-1)x_1}. \quad (4)$$

Definition 3. The q -Leibniz rule for a q -derivative of a product of two functions [10] is

$$\frac{d_q^m}{d_q x^m} (g(x)f(x)) = \sum_{k=0}^m \begin{bmatrix} m \\ k \end{bmatrix}_q \frac{d_q^{m-k}}{d_q x^{m-k}} f(q^k x) \frac{d_q^k}{d_q x^k} g(x), \quad (5)$$

and

$$\left. \frac{d_q^{m-k}}{d_q x^{m-k}} f(q^k x) \right|_{x=0} = a_{k,m-k}(q) \left. \frac{d_q^{m-k}}{d_q x^{m-k}} f(x) \right|_{x=0} \quad (6)$$

where

$$a_{k,m-k}(q) = \sum_{i=0}^k \sum_{j=0}^{m-k} (-1)^i (1-q)^i [i]_q! \begin{bmatrix} k \\ i \end{bmatrix}_q \begin{bmatrix} m-k \\ j \end{bmatrix}_q q^{\frac{i(i-1)}{2} + ij} \quad (7)$$

$a_{k,m-k}(q)$ is exists when $i+j-m+k=0$ and equal zero otherwise.

Definition 4. The q -integration is defined as [10]

$$\int_0^x f(t) d_q t = (1-q)x \sum_{i=0}^{\infty} q^i f(q^i x). \quad (8)$$

2. The homotopy preliminaries

Definition 5. Assume ψ be a function of homotopy-parameter p , then

$$D_q^m(\psi) = \left. \frac{1}{[m]_q!} \frac{\partial_q^m \psi}{\partial_q p^m} \right|_{p=0} \quad (9)$$

is called the m th-order q -homotopy derivative of ψ , where $m \geq 0$ is an integer.

Theorem 1. For homotopy-series $\psi(x, p) = \sum_{i=0}^{+\infty} u_i(x)p^i$, then

$$(a) D_q^m(\psi) = u_m \quad (10)$$

$$(b) D_q^m(p\psi) = u_{m-1} \quad (11)$$

where $m \geq 0$ is an integer.

Proof.

(a) According to Taylor's theorem [29,30], the series of ψ is given by $u_m = \left. \frac{1}{[m]_q!} \frac{\partial_q^m \psi}{\partial_q p^m} \right|_{p=0}$, which given (a) by means of the Definition 5 of $D_q^m(\psi)$.

(b) According to homotopy series $\psi(x, p) = \sum_{i=0}^{+\infty} u_i(x)p^i$, then

$$D_q^m(p\psi) = D_q^m \left(p \sum_{i=0}^{+\infty} u_i p^i \right) = D_q^m \sum_{i=0}^{+\infty} u_i p^{i+1} = \sum_{i=0}^{+\infty} u_i D_q^m(p^{i+1}),$$

$$D_q^m(p^{i+1}) = \begin{cases} 1 & \text{if } i+1-m=0 \\ 0 & \text{if } i+1-m \neq 0 \end{cases} \quad (12)$$

From (12), $D_q^m(p\psi)$ exists when $i = m-1$ and zero otherwise, then

$$D_q^m(p\psi) = u_{m-1} \quad \square$$

Theorem 2. Let L be a linear operator independent of the homotopy parameter p . For homotopy series

$$\psi = \sum_{i=0}^{+\infty} u_i p^i, \quad (13)$$

then $D_q^m(L\psi) = L[D_q^m(\psi)]$

Proof. Since L is independent of p , it holds

$$L\psi = \sum_{k=0}^{+\infty} [Lu_k] p^k, \quad (14)$$

taking m th-order q -homotopy derivative on both sides of the above expression (14) and using Theorem (1a)

$$L.H.S = D_q^m(L\psi) = D_q^m \sum_{k=0}^{+\infty} [Lu_k] p^k = L(u_m). \quad (15)$$

$$R.H.S = L \left[D_q^m(\psi) \right] = L(u_m). \quad (16)$$

From Eqs. (15) and (16) then $D_q^m(L\psi) = L[D_q^m(\psi)]$.

Theorem 3. Let a homotopy-series be

$$\psi(x, t, p) = \sum_{m=0}^{+\infty} u_m(x, t) p^m,$$

where $p \in [0, 1]$, L an auxiliary linear operator independent of p and u_0 initial solution. Then

$$D_q^m \{ (1-p)L[\psi - u_0] \} = L[u_m(x, t) - \chi_m q^{m-1} u_{m-1}(x, t)], \quad (17)$$

where the operator $D_q^m(\psi)$ is defined by (9) and χ_m is defined by

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \quad (18)$$

Proof. Since L is a linear operator independent of p , then

$$(1-p)L[\psi - u_0] = L[\psi - p\psi + u_0 p - u_0].$$

Using Theorems 1 and 2, we have

$$\begin{aligned} D_q^m \{(1-p)L[\psi - u_0]\} &= D_q^m \{L[\psi - p\psi + u_0p - u_0]\} \\ &= L \{D_q^m [\psi - p\psi + u_0p - u_0]\} \\ &= L [D_q^m (\psi) - D_q^m (p\psi) + u_0 D_q^m (p)] \\ &= L [u_m - u_{m-1} + u_0 D_q^m (p)] \end{aligned}$$

which equals to $L[u_m]$ when $m = 1$, and $L[u_m - u_{m-1}]$ when $m > 1$, respectively. Thus using the definition (18) of χ_m , it holds

$$D_q^m \{(1-p)L[\psi - u_0]\} = L[u_m(x, t) - \chi_m u_{m-1}(x, t)]. \quad \square$$

Theorem 4. Let a homotopy-series be

$$\psi = \sum_{m=0}^{+\infty} u_m(x, t) p^m, \quad (19)$$

where $p \in [0, 1]$, L is an auxiliary linear operator independent of p , N is a nonlinear operator, $u_0(x, t)$ is initial solution, \hbar the control parameter and $H(x, t)$ an auxiliary function, respectively. The zeroth-order deformation equation is given by

$$(1-p)L[\psi - u_0] = p\hbar H(x, t)N[\psi]. \quad (20)$$

The corresponding m th-order deformation equation ($m \geq 1$) reads

$$L[u_m(x, t) - \chi_m u_{m-1}(x, t)] = \hbar H(x, t) D_q^{m-1} N[\psi], \quad (21)$$

where the operator D_q^{m-1} is given by (9) and χ_m is given by (18).

Proof. Using Theorem 3, we have

$$D_q^{m-1} \{(1-p)L[\psi - u_0]\} = D_q^{m-1} \{p\hbar H(x, t)N[\psi]\} \quad (22)$$

According to Theorem 3, it holds

$$D_m \{(1-p)L[\psi - u_0]\} = L[u_m(x, t) - \chi_m u_{m-1}(x, t)]. \quad (23)$$

According to Theorem (1b), one has

$$\begin{aligned} D_q^{m-1} \{p\hbar H(x, t)N[\psi]\} &= \hbar H(x, t) D_q^{m-1} \{pN[\psi]\} \\ &= \hbar H(x, t) D_q^{m-1} \{N[\psi]\} \end{aligned} \quad (24)$$

Substituting (23) and (24) into (22), one has the m th-order deformation equation

$$L[u_m(x, t) - \chi_m u_{m-1}(x, t)] = \hbar H(x, t) D_q^{m-1} N[\psi]. \quad \square$$

Theorem 5. For an arbitrary homotopy-series (19), it holds

$$D_q^m (\psi^2) = \sum_{k=0}^m a_{k,m-k}(q) u_{m-k} u_k, \quad (25)$$

where $a_{k,m-k}(q)$ is given by (7) and $m \geq 0$ is positive integer.

Proof. According to Definition 5, Theorem (1a) and the q -Leibniz Product Law (5), it holds

$$\begin{aligned} D_q^m (\psi^2) &= \frac{1}{[m]_q!} \frac{\partial_q^m (\psi^2)}{\partial_q p^m} \Big|_{p=0} \\ &= \sum_{k=0}^m \frac{1}{[k]_q! [m-k]_q!} D_q^{m-k} (\psi(x; pq^k)) D_q^k (\psi) \\ &= \sum_{k=0}^m a_{k,m-k}(q) u_k u_{m-k}. \quad \square \end{aligned}$$

3. Analysis of method

Consider the nonlinear q -difference equation:

$$\frac{d_q^m}{d_q x^m} f(x) + g \left(f, \frac{d_q}{d_q x} f, \dots, \frac{d_q^{m-1}}{d_q x^{m-1}} f \right) - y(x) = 0, \quad (26)$$

with boundary conditions

$$\mathcal{B} \left(u, \frac{\partial u}{\partial n} \right) = 0, \quad x \in \Gamma, \quad (27)$$

where \mathcal{B} is a boundary operator and Γ is boundary of the domain Ω . The Eq. (26) can be written in the form

$$N[f(x)] = 0. \quad (28)$$

where N is a nonlinear operator, x denotes independent variable, and $f(x)$ is an unknown function. From Eq. (20), we can construct zeroth-order deformation equation as follows:

$$(1-p) \frac{d_q^m}{d_q x^m} [\psi(x, p) - f_0(x)] = p\hbar H(x) (N[\psi(x, p)]), \quad (29)$$

where $p \in [0, 1]$ denotes the so-called embedding parameter. $\hbar \neq 0$ is a control parameter, $H(x)$ is a non-zero auxiliary function, $\psi(x, p)$ is an unknown function and $f_0(x)$ is an initial solution of $f(x)$. It is obvious that when $p = 0$ and $p = 1$, Eq. (29) becomes

$$\psi(x, 0) = f_0(x), \quad \psi(x, 1) = f(x) \quad (30)$$

respectively. Thus as p increases from zero to one, the solution $\psi(x, p)$ varies from the initial solution $f_0(x)$ to the solution $f(x)$. The initial solution $f_0(x)$ satisfies the conditions (27). Expanding $\psi(x, p)$ in the Taylor series [29,30] then

$$\psi(x, p) = f_0(x) + \sum_{m=1}^{+\infty} f_m(x) p^m, \quad (31)$$

where

$$f_m(x) = D_q^m \psi(x, p) \Big|_{p=0}, \quad (32)$$

To determine the higher order terms $f_m(x) (m, 1, 2, \dots)$ define the vector

$$\vec{f}_i(x) = \{f_0(x), f_1(x), \dots, f_i(x)\} \quad (33)$$

From Theorem 4, the so-called m th-order deformation equation is

$$\frac{d_q^m}{d_q x^m} [f_m(x) - \chi_m f_{m-1}(x)] = \hbar H(x) R_m(\vec{f}_{m-1}(x)), \quad (34)$$

where

$$R_m(\vec{f}_{m-1}(x)) = \frac{1}{[m-1]_q!} \frac{\partial_q^{m-1} (N[\psi(x, p)])}{\partial_q p^{m-1}} \Big|_{p=0}, \quad (35)$$

and χ_m is given by (18). Now the solution of the m th-order deformation Eq. (34) for $m \geq 1$ when $H(x) = 1$ becomes

$$f_m(x) = \chi_m f_{m-1}(x) + \underbrace{\int \dots \int}_r \left[\hbar R_m(\bar{f}_{m-1}(x)) \right] \underbrace{d_q x d_q x \dots d_q x}_r + c_0 + c_1 x + \dots + c_{r-1} x^{r-1} \tag{36}$$

where c_0, c_1, \dots, c_{r-1} are constants and can be calculated by

$$\left. \frac{\partial_q^m \mathcal{B}(\psi(x, p), \frac{\partial \psi(x, p)}{\partial m})}{\partial_q p^m} \right|_{p=0} = 0, \tag{37}$$

Starting by $f_0(x)$, we obtain the functions $f_m(x)$ for $m, 1, 2, 3, \dots$ form Eqs. (36) and (37) successively. Accordingly, the M -th order of approximate solution of the problem (26) and (27) is given by

$$f(x) \cong F_M(x) = \sum_{m=0}^M f_m(x). \tag{38}$$

4. Numerical examples

Example 1. Suppose the nonlinear q -difference equation is [15] in the following form:

$$\frac{d_q}{d_q x} f(x) + f^2(x) - 1 = 0, \tag{39}$$

and

$$f(0) = 0. \tag{40}$$

For $q \rightarrow 1$, the problem (39) becomes

$$\frac{df(x)}{dx} + f^2(x) - 1 = 0, \tag{41}$$

and has the exact solution

$$f(x) = \tanh(x). \tag{42}$$

Now, apply the homotopy analysis method for the problems (39) and (40). Choosing the initial solution $f_0(x) = \text{zero}$, the m th-order deformation Eq. (36) for $m \geq 1$ becomes

$$f_m(x) = \chi_m f_{m-1}(x) + \int \hbar R_m(\bar{f}_{m-1}(x)) d_q x + c_0 \tag{43}$$

where c_0 can be calculated by

$$f_m(0) = 0 \tag{44}$$

and by using Theorems 1 and 5, $R_m(\bar{f}_{m-1}(x))$ is defined by

$$R_m(\bar{f}_{m-1}(x)) = \frac{d_q}{d_q x} f_{m-1}(x) + \sum_{k=0}^{m-1} a_{k,m-1-k}(q) f_k f_{m-1-k} - (1 - \chi_m), \tag{45}$$

Now giving the solution of m th-order deformation Eq. (43) as follows:

$$f_1(x) = -\hbar x,$$

$$f_2(x) = -\hbar(1 + \hbar)x,$$

$$f_3(x) = -\hbar(1 + \hbar)^2 x + \frac{\hbar^3 q x^3}{1 + q + q^2},$$

$$f_4(x) = \hbar^2(1 + \hbar)^2(2 - 3q^2 + q^3 + 2q^4 + q^5)x^2 - \frac{\hbar^4 q(1 - q^2 + q^3 + q^5)x^4}{1 + q + q^2},$$

$$f_5(x) = -\hbar(1 + \hbar)^4 x + \frac{\hbar^3(1 + \hbar)^2(3 - 2q^2 + 2q^3 + 2q^4 + q^5)x^3}{1 + q + q^2} - \frac{\hbar^5 q(1 - q^2 + q^3 + q^5)x^5}{(1 + q + q^2)(1 + q + q^2 + q^3 + q^4)},$$

and so on. The analytic approximation solution is given by

$$f(x) \cong F_M(x) = \sum_{m=0}^M f_m(x) \tag{46}$$

and $\left. \frac{d_q}{d_q x} f(x) \right|_{x=0}$ is given by [31]

$$\left. \frac{d_q}{d_q x} f(x) \right|_{x=0} \cong \left. \frac{d_q}{d_q x} F_M(x) \right|_{x=0} = \lim_{x \rightarrow 0} \frac{d_q}{d_q x} F_M(x) \tag{47}$$

The value of \hbar must be found. To find the valid region of \hbar , the \hbar curve given by $\left. \frac{d_q}{d_q x} F_5(x) \right|_{x=0}$ for different values of q is drawn in Fig. 1, which clearly indicates that the valid region of \hbar is about $-1.5 \leq \hbar \leq -0.5$. Also, it is clear from Fig. 1 that the value did not change with the value of q that is consistent with the problem, and from (39) and (40), we find that $\left. \frac{d_q}{d_q x} f(0) \right|_{x=0} = 1$.

For $q \rightarrow 1$ the q -difference Eq. (39) is converted to the ordinary differential Eq. (41). The analytic approximation solution for the problem (41) when $\hbar = -1$, becomes

$$F_M(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} + \dots, \tag{48}$$

Eq. (48) is consistent with the exact solution (42) for problem (41). This shows the accuracy of the method used.

Example 2. Suppose the q -analogue of an oscillator equation [15,16] is in the following form:

$$\frac{d_q^2}{d_q x^2} f(x) - f(x) = 0 \tag{49}$$

and

$$f(0) = 1, \left. \frac{d_q}{d_q x} f(x) \right|_{x=0} = 1, \tag{50}$$

and has the exact solution e_q^x which is the q -exponential function [15,16]. For $q \rightarrow 1$, the problem (49) becomes

$$\frac{d^2 f(x)}{d^2 x} - f(x) = 0, \tag{51}$$

and has the exact solution e^x .

Now, applying HAM for the problem (49) and (50). By choosing the initial solution $f_0(x) = 1 + x$, the m th-order deformation Eq. (36) for $m \geq 1$ becomes

$$f_m(x) = \chi_m f_{m-1}(x) + \iint \hbar R_m(\bar{f}_{m-1}(x)) d_q x d_q x + c_0 + c_1 x \tag{52}$$

where c_0 and c_1 can be calculated by

$$f_m(0) = 0 \quad \left. \frac{d_q}{d_q x} f(x) \right|_{x=0} = 0 \tag{53}$$

And $R_m(\bar{f}_{m-1}(x))$ is defined by

$$R_m(\bar{f}_{m-1}(x)) = \frac{d_q^2}{d_q^2 x} f_{m-1}(x) - f_{m-1}(x) \tag{54}$$

Now giving the solution of m th-order deformation Eq. (52) as follows:

$$f_1(x) = -\frac{\hbar x^2}{1+q} - \frac{\hbar x^3}{(1+q)(1+q+q^2)},$$

$$f_2(x) = -\frac{\hbar(1+\hbar)x^2}{1+q} - \frac{x^3\hbar(1+\hbar)}{(1+q)(1+q+q^2)} + \frac{x^4\hbar^2}{(1+q)^2(1+q^2)(1+q+q^2)} + \frac{x^5\hbar^2}{(1+q)^2(1+q^2)(1+q+q^2)(1+q+q^2+q^3+q^4)}$$

$$f_3(x) = -\frac{x^2\hbar(1+\hbar)^2}{1+q} - \frac{x^3\hbar(1+\hbar)^2}{(1+q)(1+q+q^2)} + \frac{2x^4\hbar^2(1+\hbar)}{(1+q)^2(1+q^2)(1+q+q^2)} + \frac{2x^5\hbar^2(1+\hbar)}{(1+q)^2(1+q^2)(1+q+q^2)(1+q+q^2+q^3+q^4)} - \frac{x^6\hbar^3}{(1+q)^3(1+q+q^2)^2(1+2q^2+q^3+2q^4+q^5+2q^6+q^8)} - \frac{x^7\hbar^3}{(1+q)^3(1+q^2)(1-q+q^2)(1+q+q^2)^2(1+q+q^2+q^3+q^4)(1+q+q^2+q^3+q^4+q^5+q^6)}$$

and so on. The analytic approximation solution is given by (38). A proper value of \hbar must be found. To find the valid region of \hbar , the \hbar curve given by $\left. \frac{d_q}{d_q x} F_6(x) \right|_{x=0}$ for $q = \frac{1}{2}$ and $q \rightarrow 1$ is drawn in Fig. 2, which clearly indicates that the valid region of \hbar is about $-1.5 \leq \hbar \leq -0.5$. When $\hbar = -1$, the analytic approximation solution becomes

$$F_M(x) = 1 + x + \frac{x^2}{1+q} + \frac{x^3}{(1+q)(1+q+q^2)} + \frac{x^4}{(1+q)^2(1+q^2)(1+q+q^2)} + \frac{x^5}{(1+q)^2(1+q^2)(1+q+q^2)(1+q+q^2+q^3+q^4)} + \dots,$$

$$F_M(x) = 1 + \frac{x}{[1]_q!} + \frac{x^2}{[2]_q!} + \frac{x^3}{[3]_q!} + \frac{x^4}{[4]_q!} + \frac{x^5}{[5]_q!} + \dots = \sum_{i=0}^M \frac{x^i}{[i]_q!} \tag{55}$$

When $f(x) = \lim_{M \rightarrow \infty} F_M(x) = e_q^x$ is an exact solution of (49). The same result can be obtained by using the variational iteration method [15,16]. For $q \rightarrow 1$, $[i]_q! = i!$, then Eq. (55)

becomes, $F_M(x) = \sum_{i=0}^M \frac{x^i}{i!}$. The limit $f(x) = \lim_{M \rightarrow \infty} \sum_{i=0}^M \frac{x^i}{i!} = e^x$ is an exact solution of (51).

Example 3. Suppose q -diffusion equation [10] is in the following form:

$$\frac{\partial_q}{\partial_q t} f(x, t) = \frac{\partial_q^2}{\partial_q x^2} f(x, t) \tag{56}$$

subjecting to the initial condition

$$f(x, 0) = e_q^x, \tag{57}$$

Eq. (56) allows the q -exponential distribution which describes a non-equilibrium system. Applying HAM for the problem (56) and (57) using the initial solution $f_0(x, t) = e_q^x$, the m th-order deformation Eq. (36) for $m \geq 1$ becomes

$$f_m(x, t) = \chi_m f_{m-1}(x, t) + \int \hbar R_m(\bar{f}_{m-1}(x, t)) d_q t + c_0 \tag{58}$$

where c_0 can be calculated by

$$f_m(x, 0) = 0 \tag{59}$$

and

$$R_m(\bar{f}_{m-1}(x, t)) = \frac{\partial_q}{\partial_q t} f_{m-1}(x, t) - \frac{\partial_q^2}{\partial_q x^2} f_{m-1}(x, t) \tag{60}$$

The solutions of m th-order deformation Eq. (58) are as follows:

$$f_1(x, t) = -e_q^x t \hbar,$$

$$f_2(x, t) = e_q^x \left(-t \hbar (1 + \hbar) + \frac{t^2 \hbar^2}{1 + q} \right),$$

$$f_3(x, t) = e_q^x \left(-t \hbar (1 + \hbar)^2 + \frac{2t^2 \hbar^2 (1 + \hbar)}{1 + q} - \frac{t^3 \hbar^3}{(1 + q)(1 + q + q^2)} \right),$$

$$f_4(x, t) = e_q^x \left(-t \hbar (1 + \hbar)^3 + \frac{3t^2 \hbar^2 (1 + \hbar)^2}{1 + q} - \frac{3t^3 \hbar^3 (1 + \hbar)}{(1 + q)(1 + q + q^2)} + \frac{t^4 \hbar^4}{(1 + q)^2 (1 + q^2) (1 + q + q^2)} \right)$$

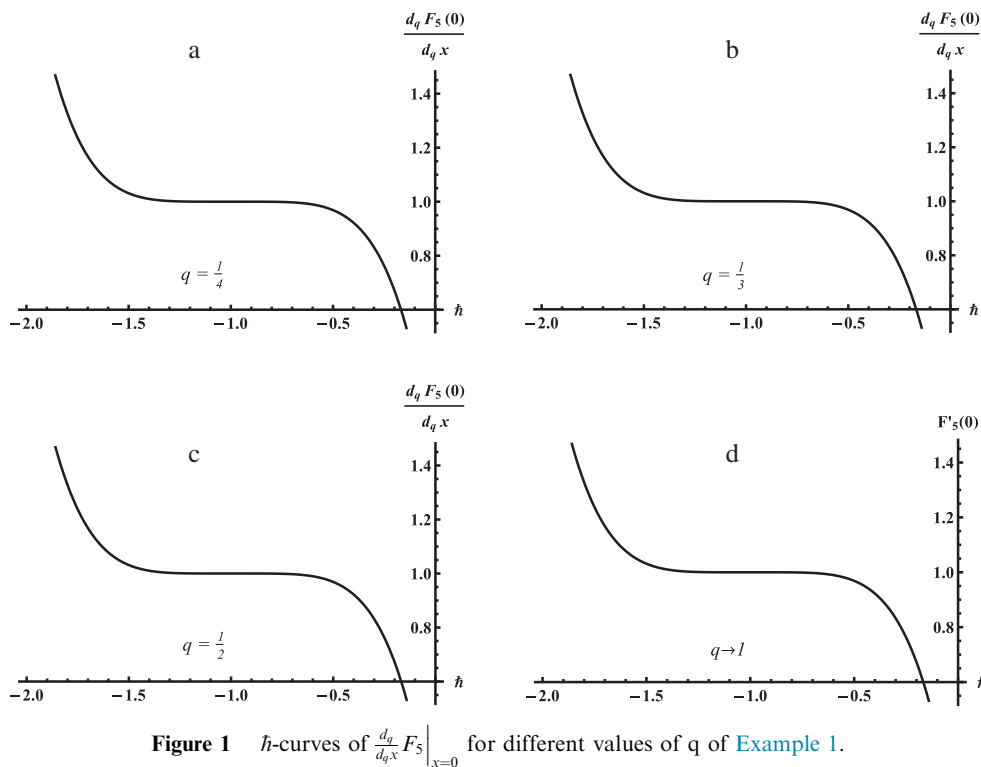


Figure 1 \hbar -curves of $\frac{d_q F_5(0)}{d_q x} \Big|_{x=0}$ for different values of q of Example 1.

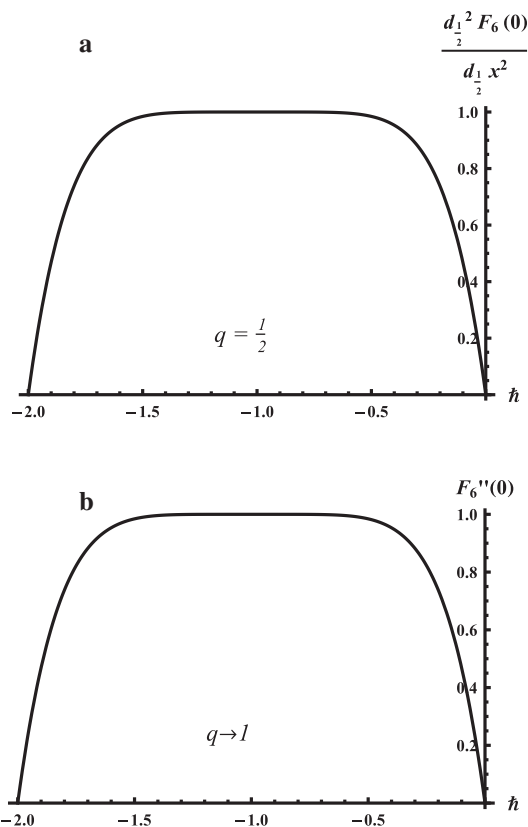


Figure 2 \hbar -curves of $\frac{d_q^2 F_6(x)}{d_q x^2} \Big|_{x=0}$ for different values of q of Example 2.

and so on. The analytic approximation solution is

$$\begin{aligned}
 F_M(x, t) = & \sum_{m=0}^M f_m(x, t) = e_q^x - e_q^x t \hbar + e_q^x \left(-t \hbar (1 + \hbar) + \frac{t^2 \hbar^2}{1 + q} \right) \\
 & + e_q^x \left(-t \hbar (1 + \hbar)^2 + \frac{2t^2 \hbar^2 (1 + \hbar)}{1 + q} \right. \\
 & \left. - \frac{t^3 \hbar^3}{(1 + q)(1 + q + q^2)} \right) + \dots
 \end{aligned} \tag{61}$$

For $\hbar = -1$, Eq. (61) becomes

$$\begin{aligned}
 F_M(x, t) = & e_q^x \left(1 + t + \frac{t^2}{1 + q} + \frac{t^3}{(1 + q)(1 + q + q^2)} \right. \\
 & \left. + \frac{t^4}{(1 + q)^2 (1 + q^2)(1 + q + q^2)} + \dots \right) \\
 = & e_q^x \left(1 + \frac{t}{[1]_q!} + \frac{t^2}{[2]_q!} + \frac{t^3}{[3]_q!} + \frac{t^4}{[4]_q!} + \dots \right) \\
 = & e_q^x \sum_{i=0}^M \frac{t^i}{[i]_q!}
 \end{aligned} \tag{62}$$

The limit $f(x, t) = \lim_{M \rightarrow \infty} F_M(x, t) = e_q^x e_q^t$ is an exact solution of (56). The same result can be obtained by using the q -differential transform [10].

5. Conclusion

In this study, the homotopy analysis method is successfully extended to q -differential equations. The q -diffusion equation and some examples are analytically investigated to show the

efficiency and the importance of the proposed method. The results demonstrate reliability and efficiency of the algorithm developed.

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